LINEAR PROGRAMMING

Graphical Method

Terminology

- Solution: The set of decision variables x_j (j = 1,2,3,…,n) that satisfy the constraints of an LP problem.

- Feasible Solution: The set of values of decision variables x_j (j = 1,2,3,…,n) that satisfy all the constraints and non-negativity conditions of an LP problem.

- Basic Solution: For a set of *m* simultaneous equations in *n* variables (*n*>*m*) in LP problem, a solution obtained by setting (*n*-*m*) variables equal to zero and solving for remaining m equations in *m* variables is called basic solution.

Terminology

- Basic feasible solution: A feasible solution to an LP problem which is also the basic solution.
  - *Degenerate*: If the value of at least one basic variable is zero.
  - *Non Degenerate*: If value of all m basic variables is non-zero and +ve.
- Optimum basic feasible solution: A basic feasible solution that optimizes the objective function value.
- Unbounded Solution: A solution that can increase or decrease infinitely the value of the objective function.

Graphical Solution of LPP

- The collection of all feasible solutions to an LP Problem constitutes a convex set whose extreme points correspond to the basic feasible solutions.

- There are a finite number of basic feasible solutions within the feasible solution space.

- If the convex set of the feasible solutions of the system of simultaneous equations: $Ax = b$, $x \geq 0$, is a convex polyhedron, then at least one of the extreme points gives an optimal solution.

- If the optimal solution occurs at more than one extreme points, the value of objective function will be the same for all convex combinations of these extreme points.

Graphical Method: Extreme point method

- Formulate the objective function (*max/min*)
- Plot the constraints on graph paper by removing the inequality sign
- Determine the feasible region with reference to origin as per inequality sign
- Shade the common area which satisfy all the constraints together
- Locate the coordinates of all the vertex points of polyhedron formed by intersection of constraints
- Check the values of objective function for each vertex point and accordingly decide the optimum

Example1: A Simple Maximization Problem

$$\begin{aligned}
\text{Max} \quad & 5x_1 + 7x_2 \\
\text{s.t.} \quad & x_1 \leq 6 \\
& 2x_1 + 3x_2 \leq 19 \\
& x_1 + x_2 \leq 8 \\
& x_1 \geq 0 \text{ and } x_2 \geq 0
\end{aligned}$$

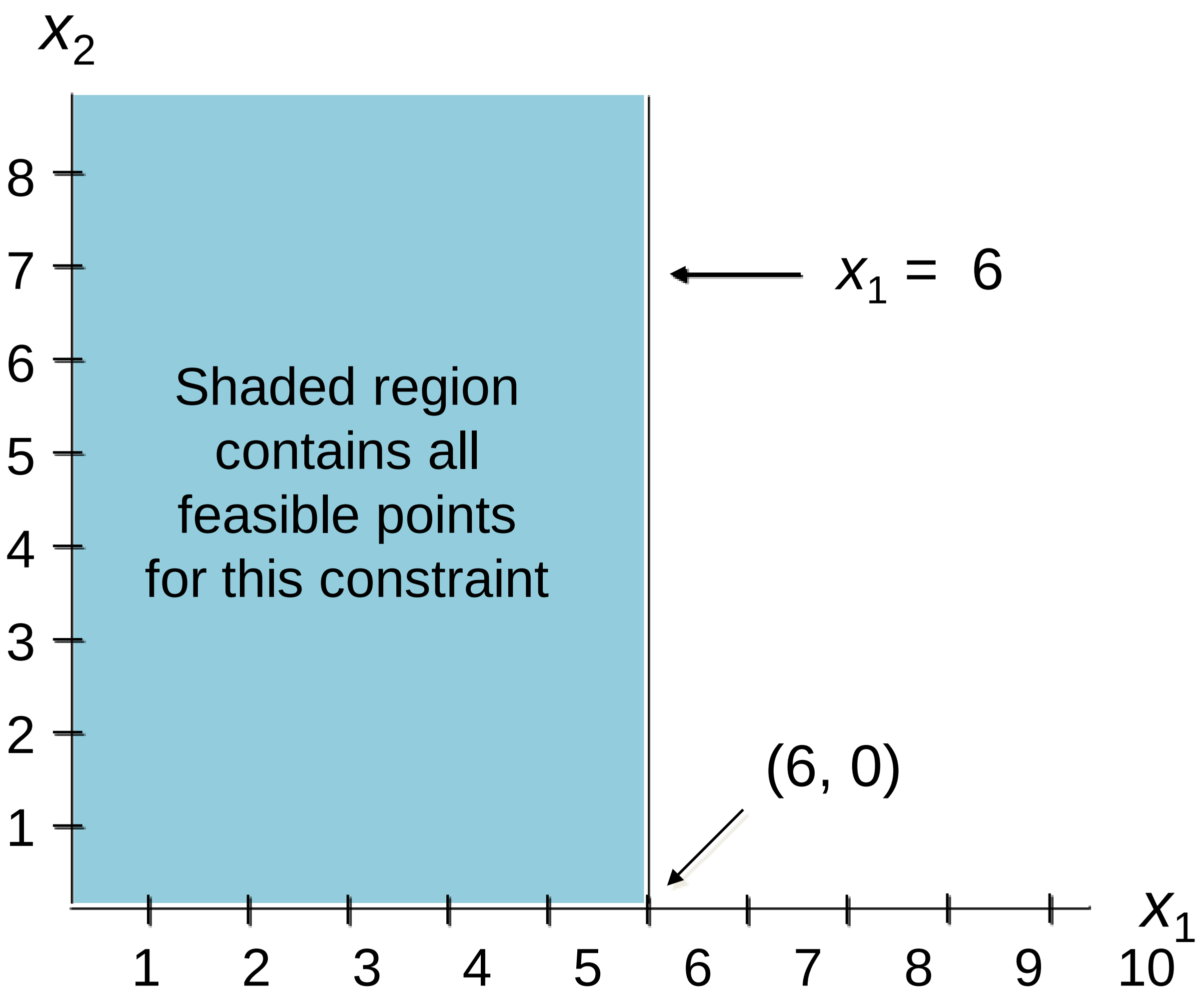
Objective Function

Regular Constraints

Non-negativity Constraints

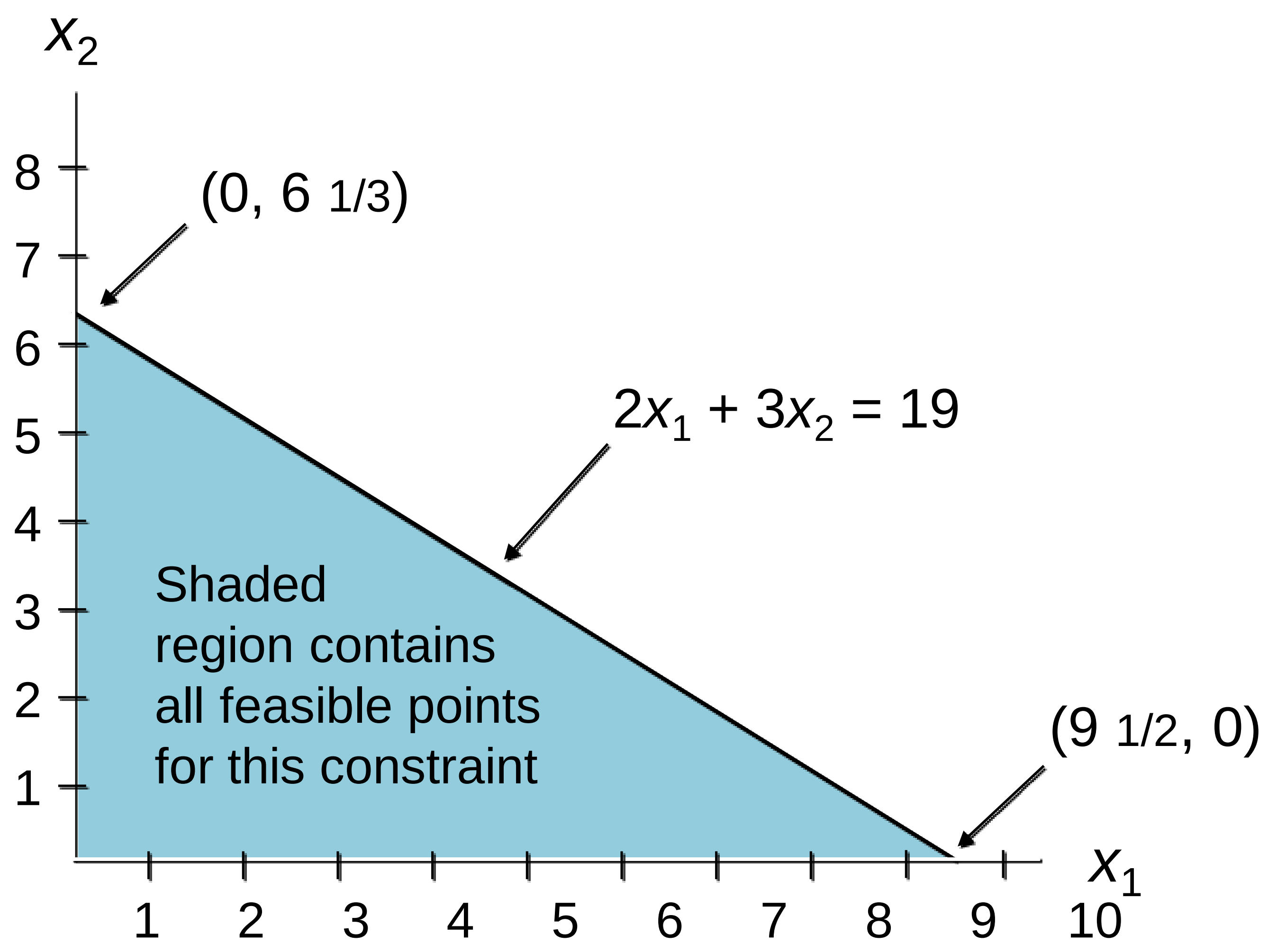
Example 1: Graphical Solution

First Constraint Graphed



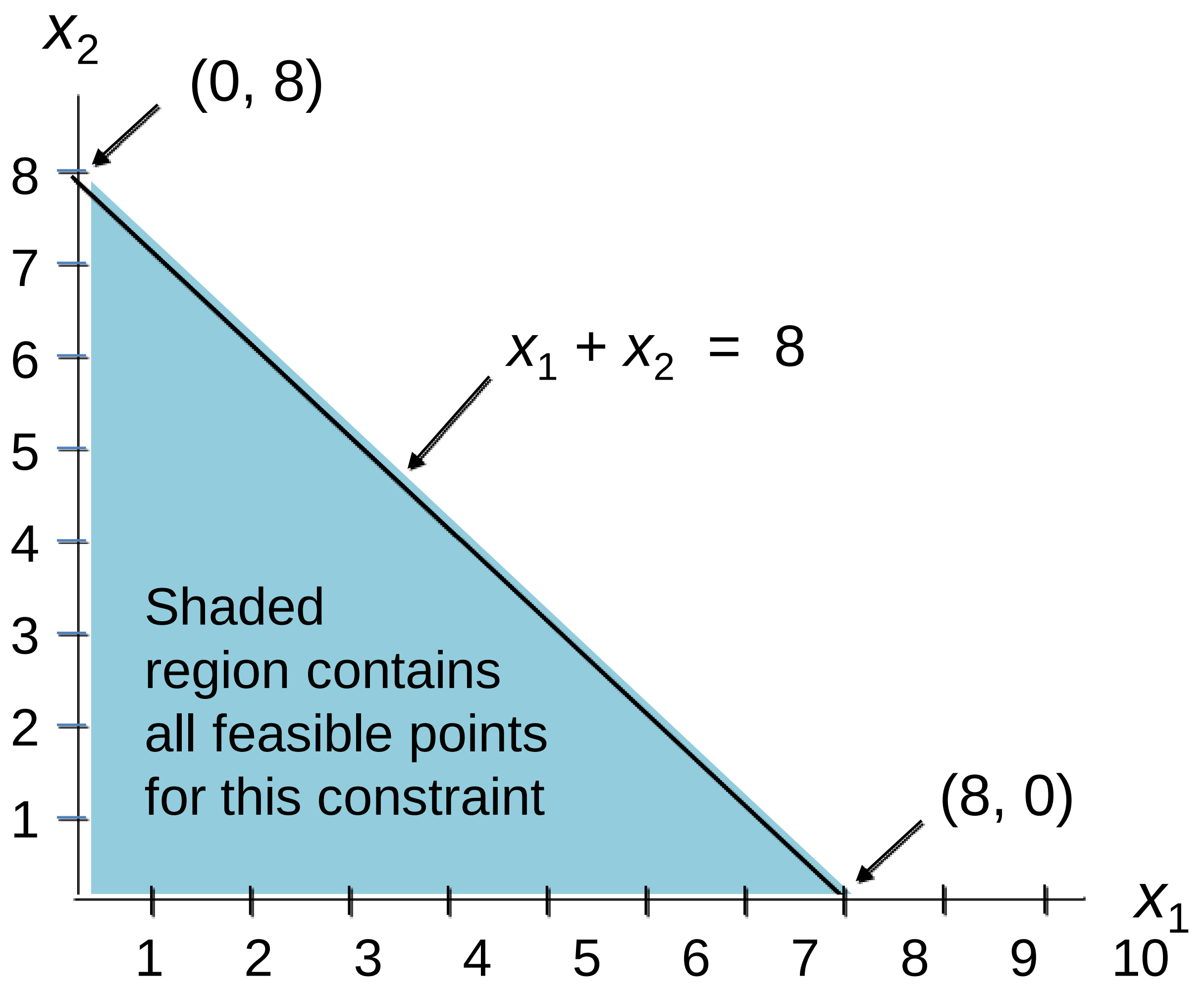
Example 1: Graphical Solution

Second Constraint Graphed



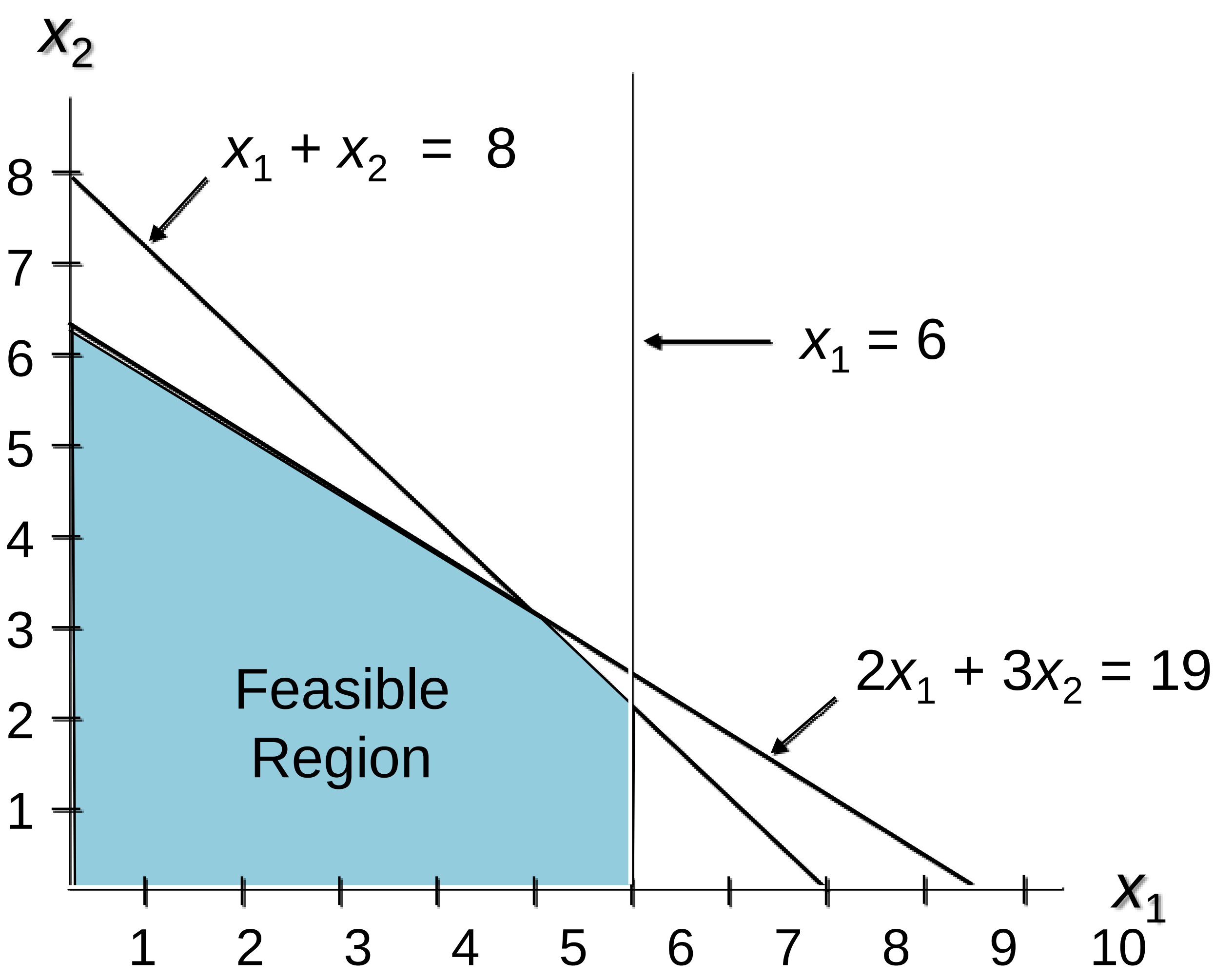
Example 1: Graphical Solution

Third Constraint Graphed

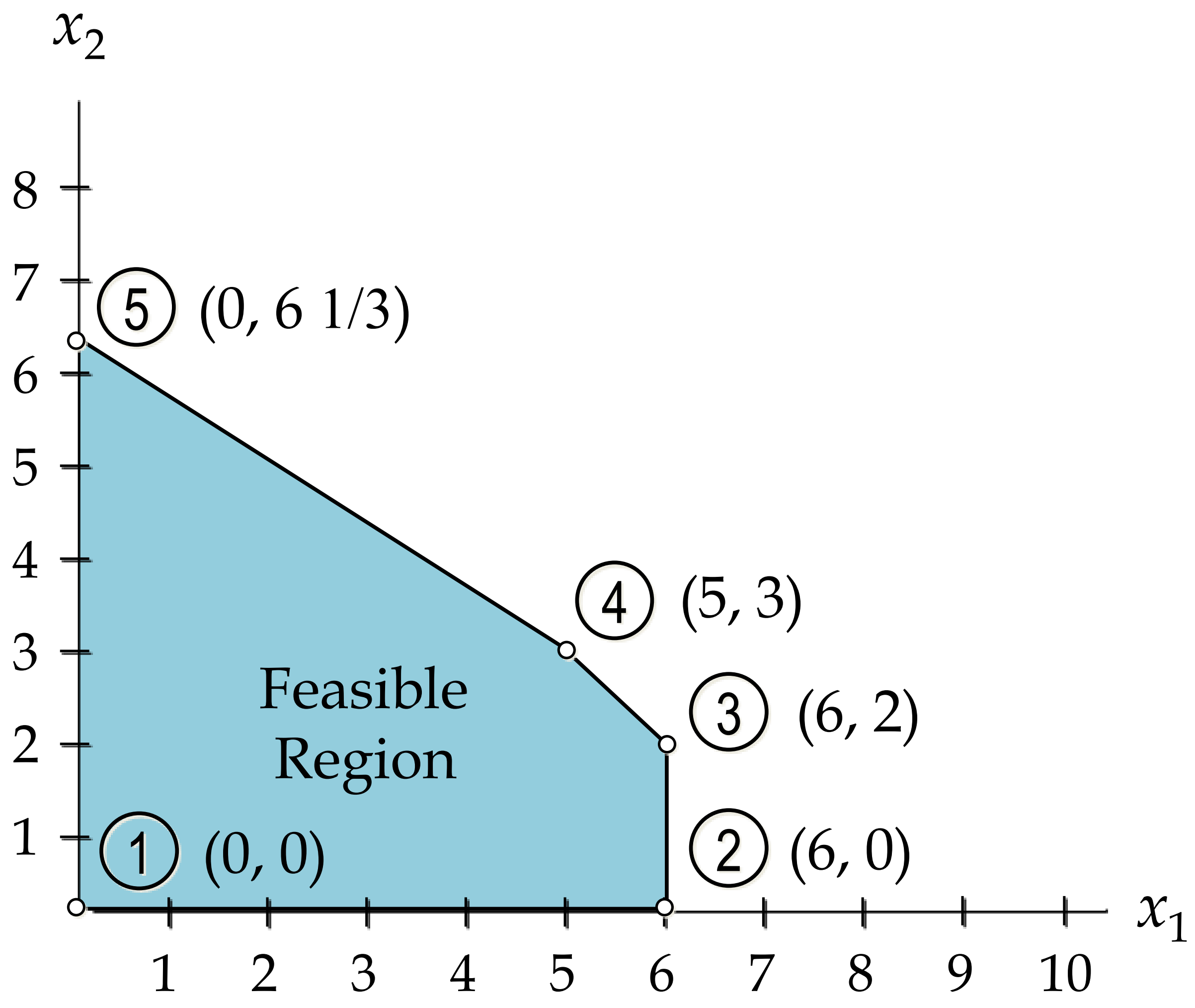


Example 1: Graphical Solution

Combined-Constraint Graph Showing Feasible Region



Example 1: Extreme Points

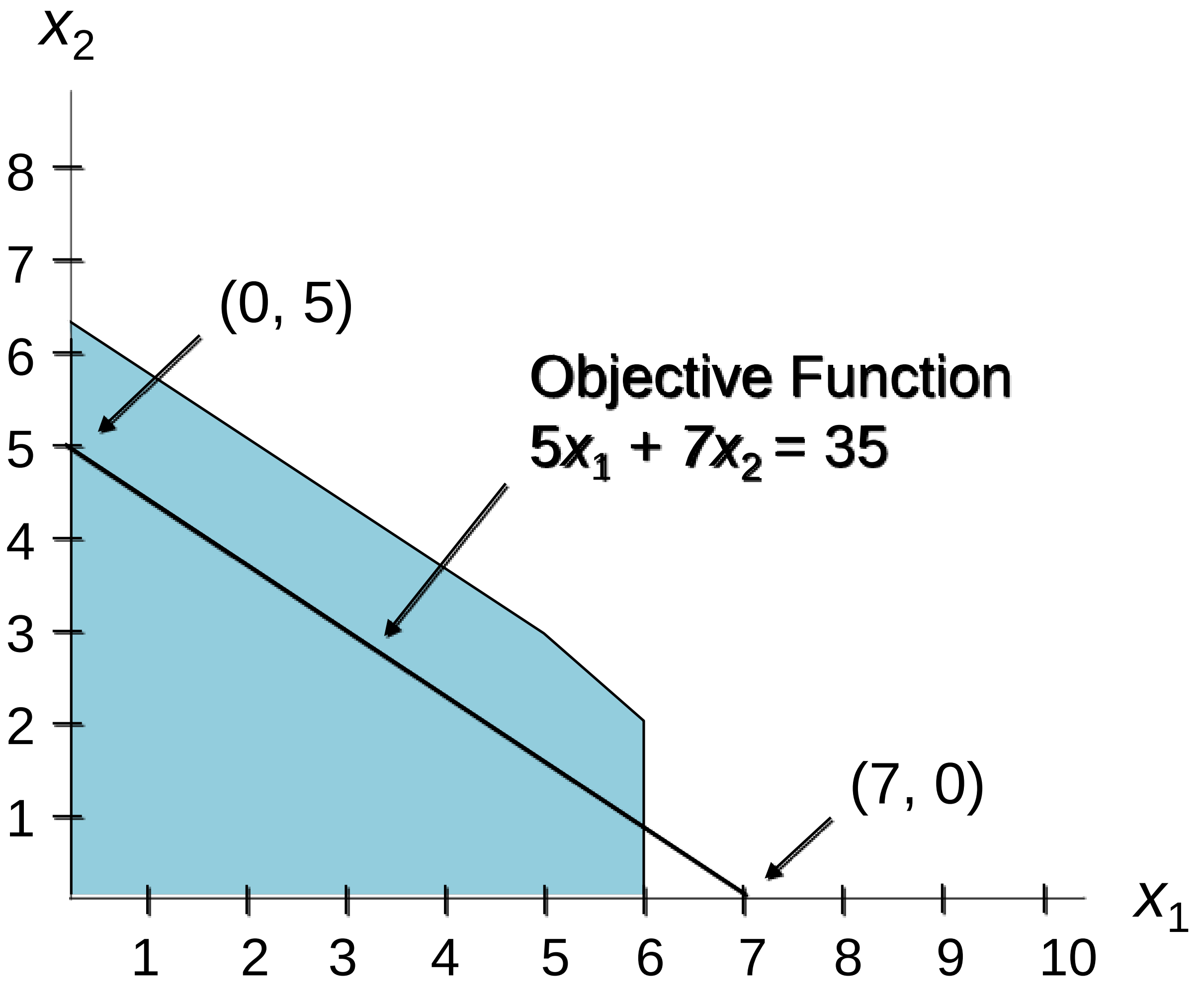


Iso-profit (cost) Function line method

- Prepare a graph of the feasible solutions for each of the constraints.
- Determine the feasible region that satisfies all the constraints simultaneously.
- Draw an objective function line by arbitrarily taking z value.
- Move parallel objective function lines toward larger (smaller) objective function values without entirely leaving the feasible region.
- Any feasible solution on the objective function line with the largest (smallest) value is an optimal solution.

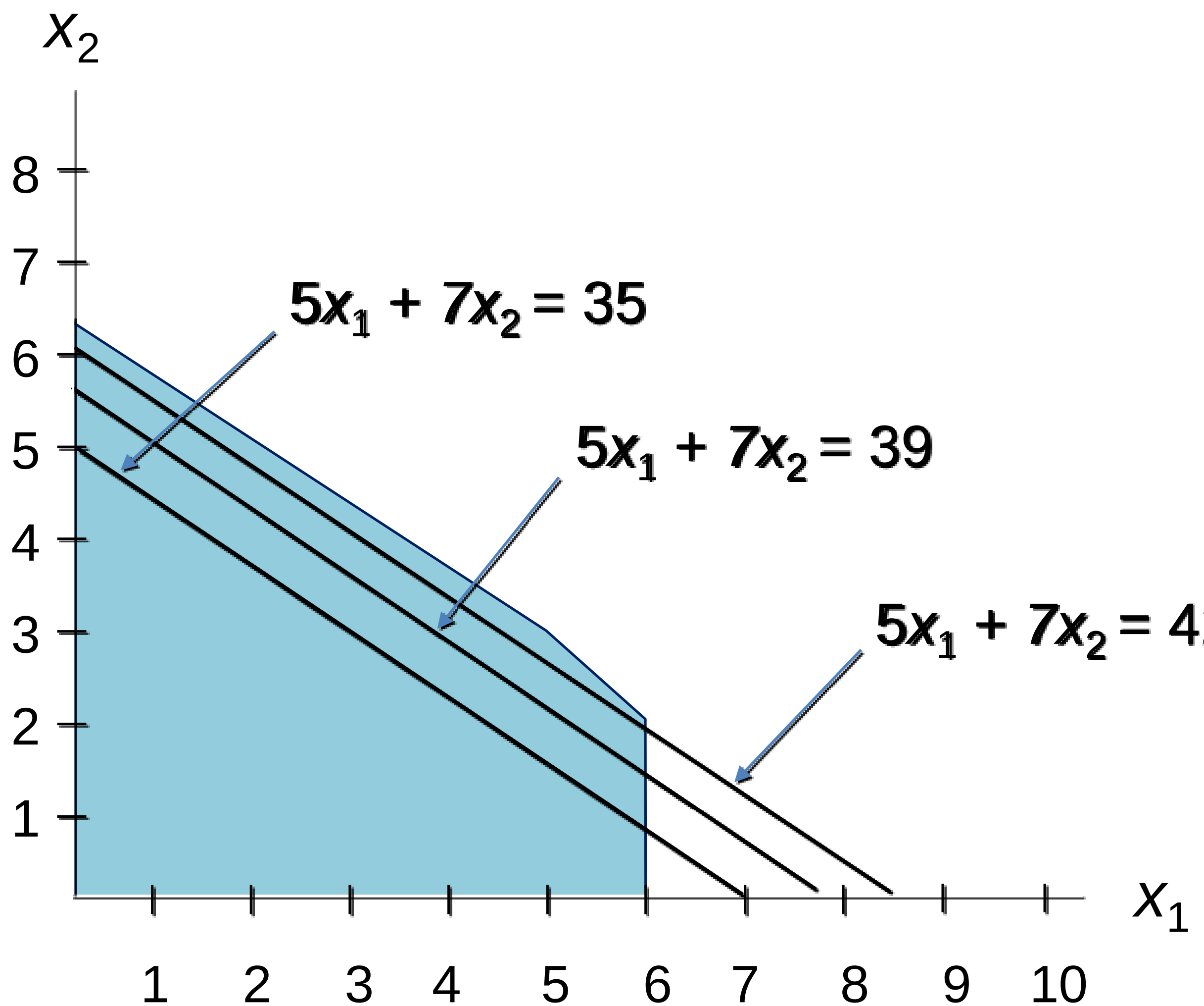
Example 1: Graphical Solution

Objective Function Line



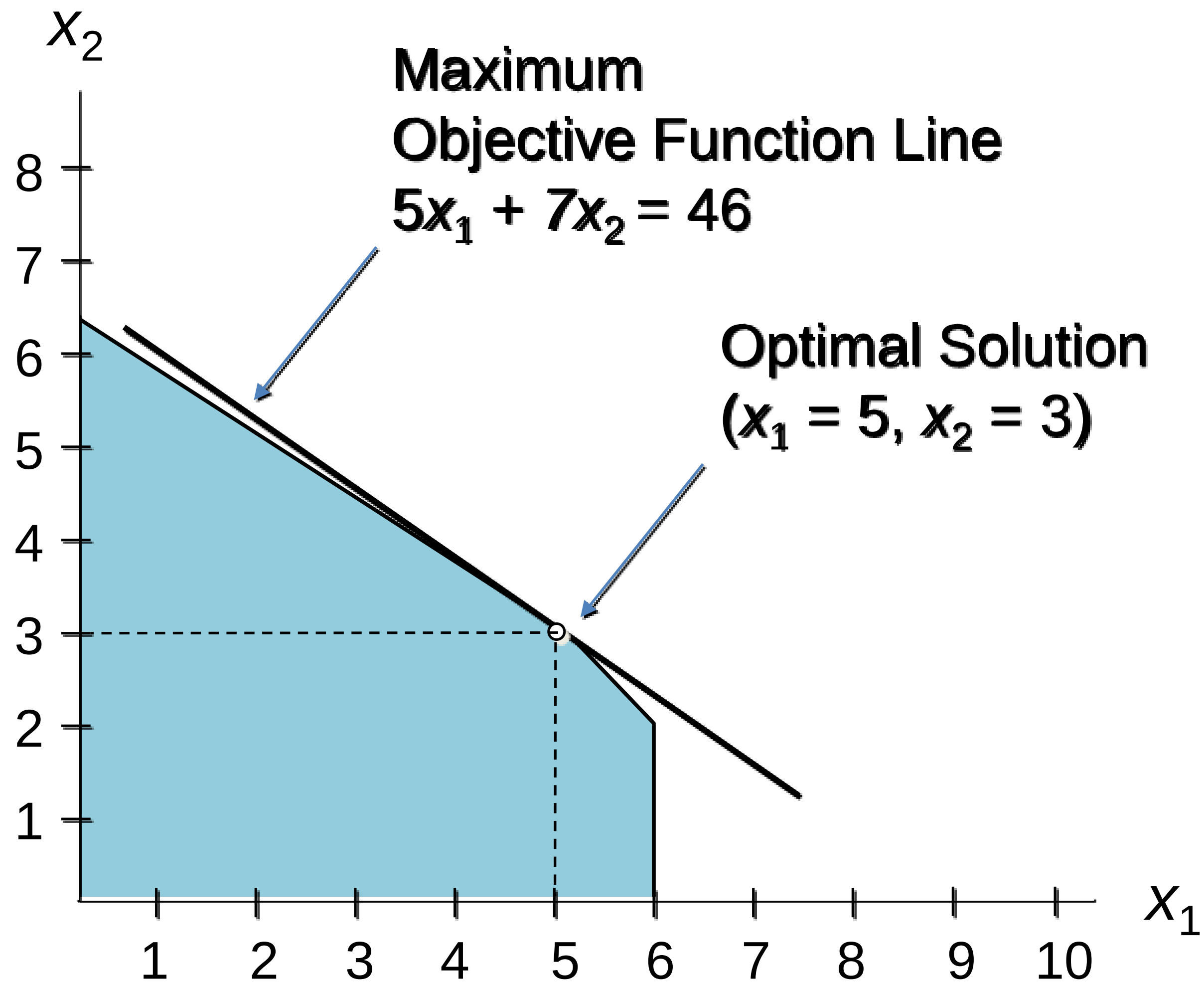
Example 1: Graphical Solution

Selected Objective Function Lines



Example 1: Graphical Solution

Optimal Solution



Example 2: A Simple Minimization Problem

LP Formulation

$$\begin{aligned}
\text{Min} \quad & 5x_1 + 2x_2 \\
\text{s.t.} \quad & 2x_1 + 5x_2 \geq 10 \\
& 4x_1 - x_2 \geq 12 \\
& x_1 + x_2 \geq 4 \\
& x_1, x_2 \geq 0
\end{aligned}$$

Example 2: Graphical Solution

Graph the Constraints

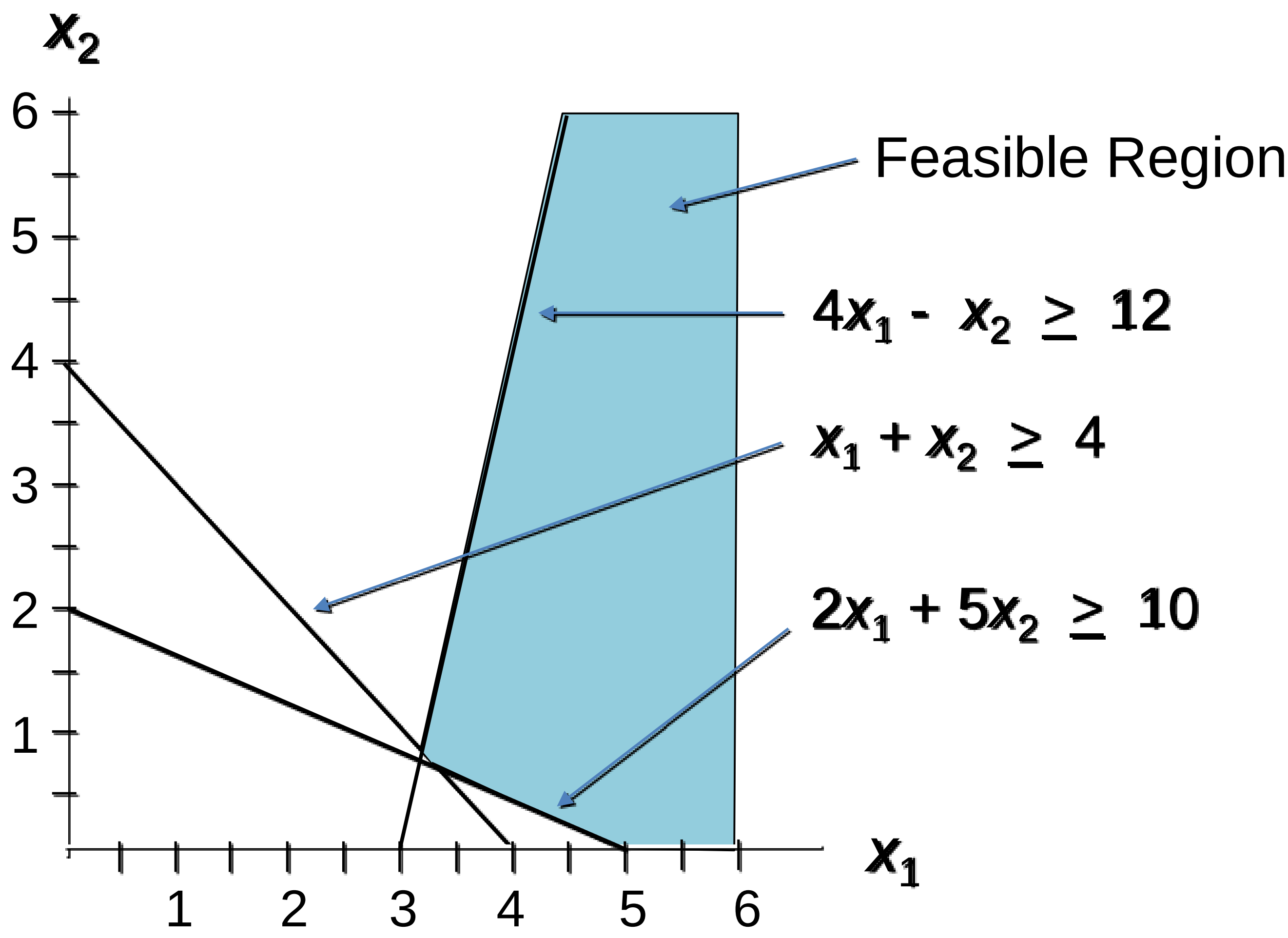
Constraint 1: When $x_1 = 0$, then $x_2 = 2$; when $x_2 = 0$, then $x_1 = 5$. Connect (5,0) and (0,2). The ">" side is above this line.

Constraint 2: When $x_2 = 0$, then $x_1 = 3$. But setting x_1 to 0 will yield $x_2 = -12$, which is not on the graph. Thus, to get a second point on this line, set x_1 to any number larger than 3 and solve for x_2: when $x_1 = 5$, then $x_2 = 8$. Connect (3,0) and (5,8). The ">" side is to the right.

Constraint 3: When $x_1 = 0$, then $x_2 = 4$; when $x_2 = 0$, then $x_1 = 4$. Connect (4,0) and (0,4). The ">" side is above this line.

Example 2: Graphical Solution

- Constraints Graphed



Example 2: Graphical Solution

- Graph the Objective Function

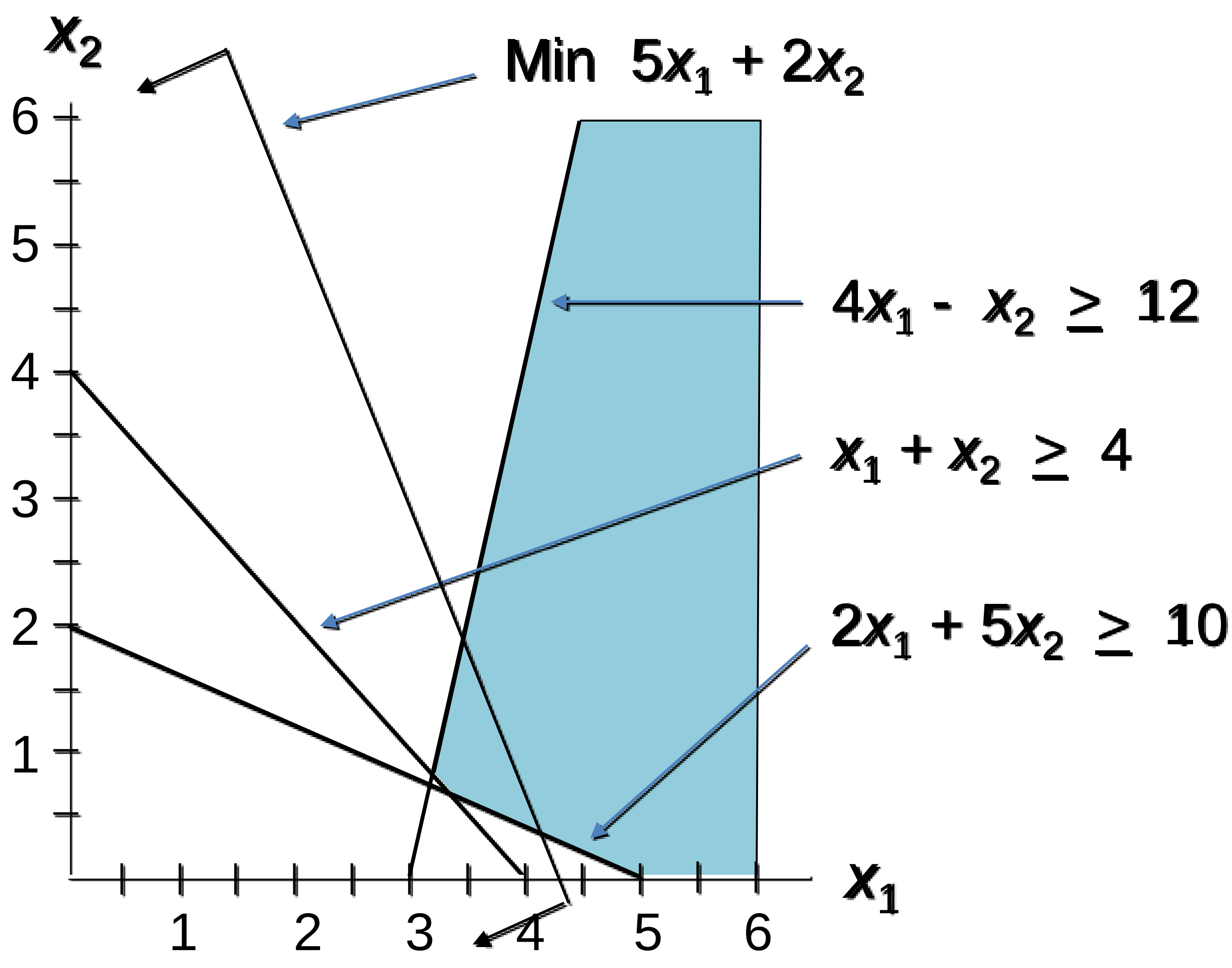
  Set the objective function equal to an arbitrary constant (say 20) and graph it. For $5x_1 + 2x_2 = 20$, when $x_1 = 0$, then $x_2 = 10$; when $x_2 = 0$, then $x_1 = 4$. Connect (4,0) and (0,10).

- Move the Objective Function Line Toward Optimality

  Move it in the direction which lowers its value (down), since we are minimizing, until it touches the last point of the feasible region, determined by the last two constraints.

Example 2: Graphical Solution

Objective Function Graphed



Example 2: Graphical Solution

- Solve for the Extreme Point at the Intersection of the Two Binding Constraints

$$4x_1 - x_2 = 12$$

$$x_1 + x_2 = 4$$

Adding these two equations gives:

$$5x_1 = 16 \quad \text{or} \quad x_1 = 16/5$$

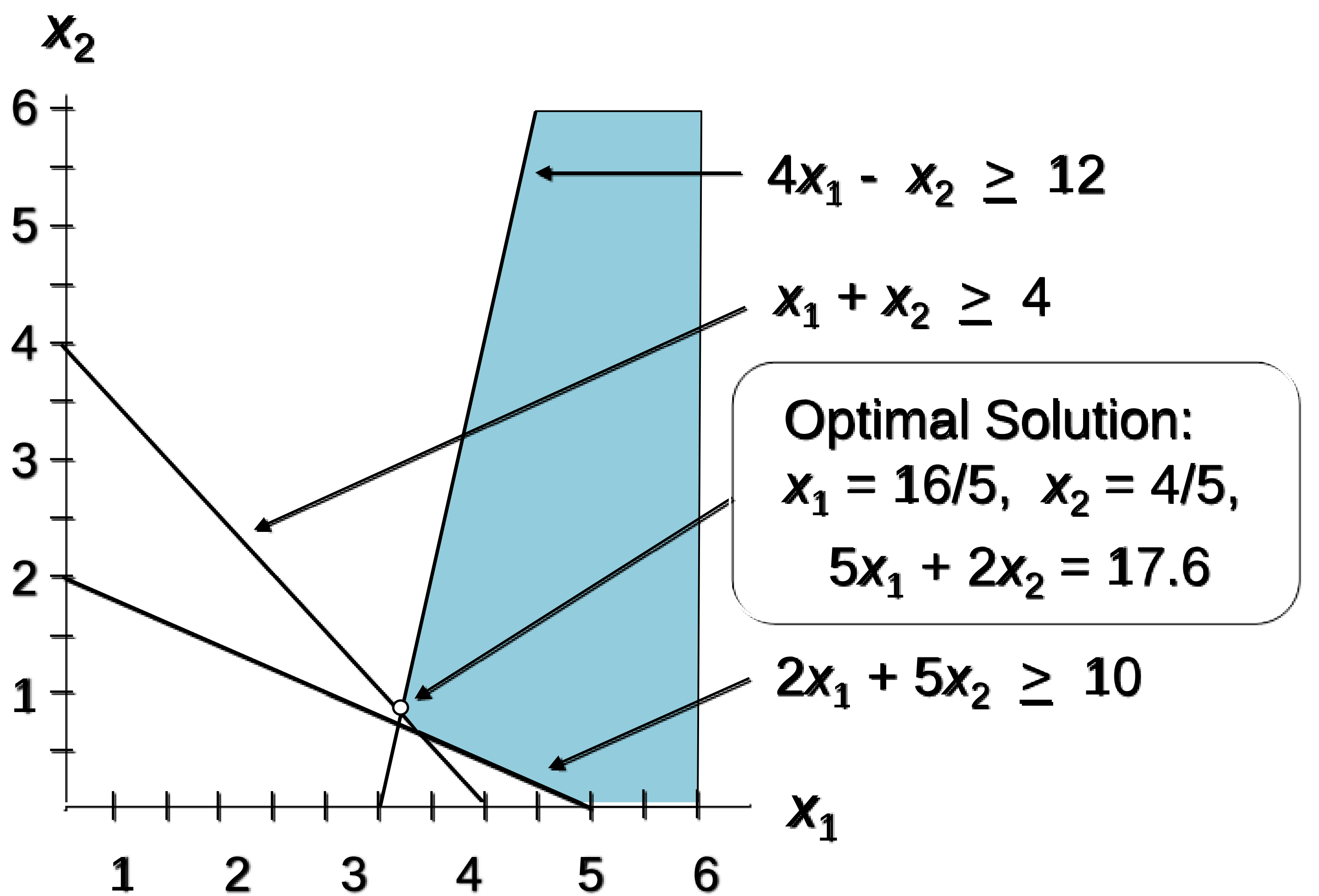
Substituting this into $x_1 + x_2 = 4$ gives: $x_2 = 4/5$

Solve for the Optimal Value of the Objective Function

$$5x_1 + 2x_2 = 5(16/5) + 2(4/5) = 88/5$$

Example 2: Graphical Solution

Optimal Solution



Special Cases

Infeasibility

- No solution to the LP problem satisfies all the constraints, including the non-negativity conditions.
- Graphically, this means a feasible region does not exist.
- Causes include:
  - A formulation error has been made.
  - Management’s expectations are too high.
  - Too many restrictions have been placed on the problem (i.e. the problem is over-constrained).

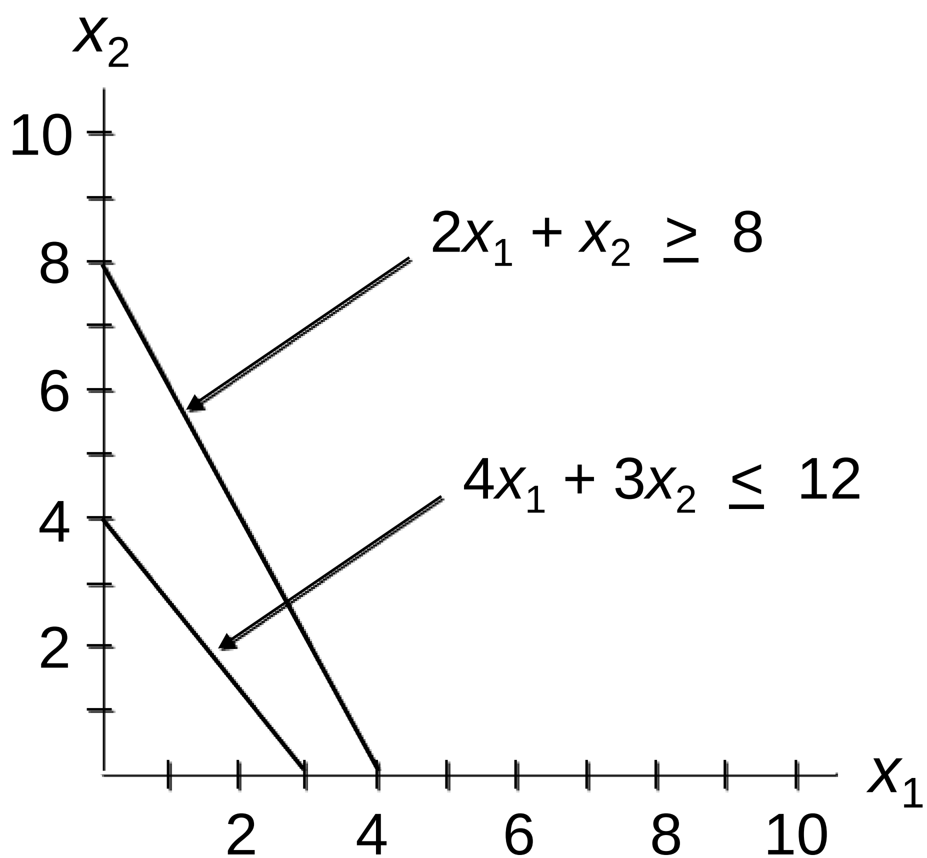
Example: Infeasible Problem

Consider the following LP problem

$$
\begin{array}{lrcr}
\text{Max} & 2x_1 + 6x_2 & & \\
\text{s.t.} & 4x_1 + 3x_2 & \leq & 12 \\
 & 2x_1 + x_2 & \geq & 8 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$

Example: Infeasible Problem

There are no points that satisfy both constraints, so there is no feasible region (and no feasible solution)



Special Cases

Unboundedded

- The solution to a maximization LP problem is unbounded if the value of the solution may be made indefinitely large without violating any of the constraints.
- For real problems, this is the result of improper formulation. (Quite likely, a constraint has been mistakenly omitted.)

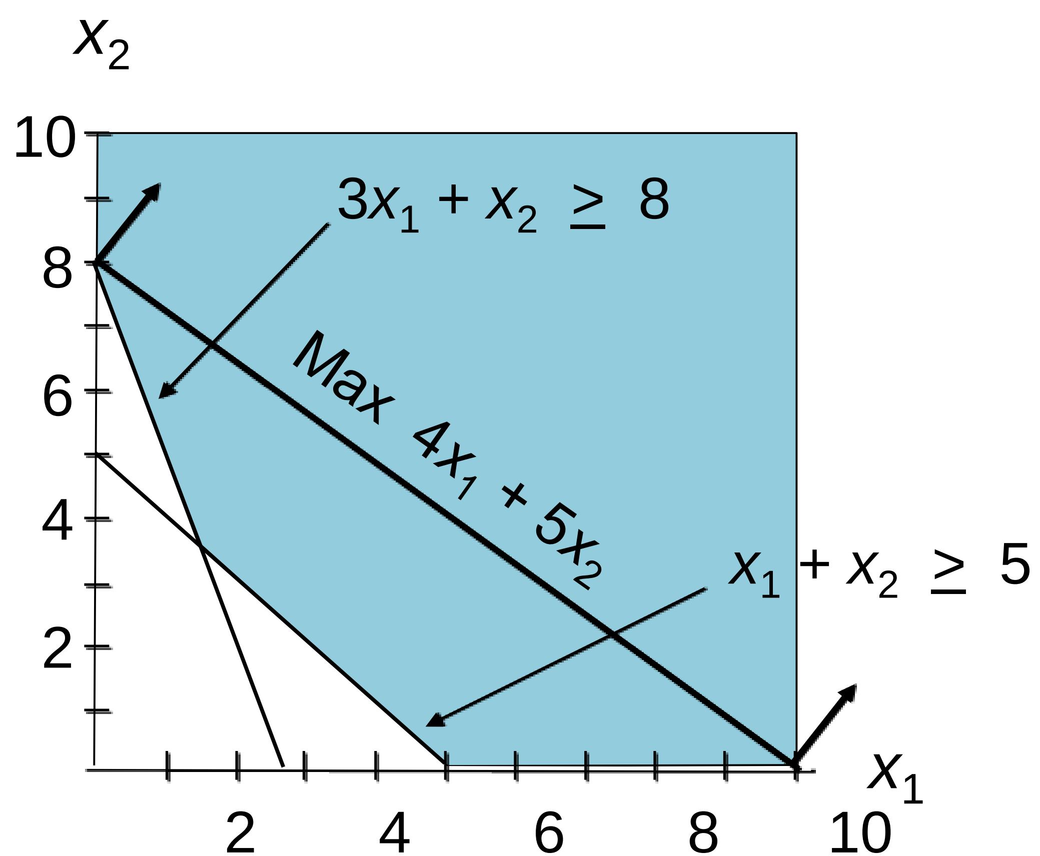
Example: Unbounded Solution

Consider the following LP problem

$$
\begin{aligned}
\text{Max} \quad & 4x_1 + 5x_2 \\
\text{s.t.} \quad & x_1 + x_2 \geq 5 \\
& 3x_1 + x_2 \geq 8 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Example: Unbounded Solution

The feasible region is unbounded and the objective function line can be moved outward from the origin without bound, infinitely increasing the objective function.



Problem-1

Solve the given LPP

$$\text{Max} \quad Z = 100x_1 + 40x_2$$

$$\begin{aligned}
\text{s.t.} \quad 10x_1 + 4x_2 &\leq 2000 \\
3x_1 + 2x_2 &\leq 900 \\
6x_1 + 12x_2 &\leq 3000 \\
x_1, x_2 &\geq 0
\end{aligned}$$

Problem-2

Solve the given LPP

Max $Z = 2x_1 + 3x_2$

s.t. $x_1 - x_2 \leq 2$

$x_1 + x_2 \geq 4$

$x_1, x_2 \geq 0$

Problem-3

Solve the given LPP

Max $Z = 4x_1 + 3x_2$

s.t. $x_1 - x_2 \leq -1$

$-x_1 + x_2 \leq 0$

$x_1, x_2 \geq 0$

Problem-4

Solve the given LPP

$$\text{Min} \quad Z = -x_1 + 2x_2$$

$$\begin{aligned} \text{s.t.} \quad & -x_1 + 3x_2 \le 10 \\ & x_1 + x_2 \le 6 \\ & x_1 - x_2 \le 2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Problem-5

Solve the given LPP

Max $Z = 2x_1 + 3x_2$

s.t. $x_1 + x_2 \leq 30$

$x_2 \geq 3$

$0 \leq x_2 \leq 12$

$0 \leq x_1 \leq 20$

$x_1 - x_2 \geq 0$

$x_1, x_2 \geq 0$